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Quality adjusted GEKS-type indices for price comparisons based on scanner data

Jacek Białek[1]

Abstract

A wide variety of retailers (supermarkets, home electronics, Internet shops, etc.) provide scanner data containing information at the level of the barcode, e.g. the Global Trade Item Number (GTIN). As scanner data provide complete transaction information, we may use the expenditure shares of items as weights for calculating price indices at the lowest (elementary) level of data aggregation. The challenge here is the choice of the index formula which should be able to reduce chain drift bias and substitution bias. Multilateral index methods seem to be the best choice due to the dynamic character of scanner data. These indices work on a whole-time window and are transitive, which is key to the elimination of the chain drift effect. Following what is called an identity test, however, it may be expected that even when only prices return to their original values, the index becomes one. Unfortunately, the commonly used multilateral indices (GEKS, CCDI, GK, TPD, TDH) do not meet the identity test. The paper discusses the proposal of two multilateral indices and their weighted versions. On the one hand, the design of the proposed indices is based on the idea of the GEKS index. On the other hand, similarly to the Geary-Khamis method, it requires quality adjusting. It is shown that the proposed indices meet the identity test and most other tests. In an empirical and simulation study, these indices are compared with the SPQ index, which is relatively new and also meets the identity test. The analytical considerations as well as empirical studies confirm the high usefulness of the proposed indices.

Key words: scanner data, product classification, product matching, Consumer Price Index, multilateral indices, GEKS index.

1. Introduction

Scanner data have numerous advantages compared to traditional survey data collection because such data sets are much bigger than traditional ones and they contain complete transaction information, i.e. information about prices and quantities at the lowest COICOP (Classification of Individual Consumption by Purpose) level. Scanner data contain expenditure information at the item level (i.e. at the retailer's code or the Global Trade Article Number (GTIN) / European Article number (EAN) / Stock Keeping Unit (SKU) barcode level), which makes it possible to use expenditure shares of items as weights for calculating price indices at the lowest (elementary) level of data aggregation. Most statistical agencies use bilateral index numbers in the CPI measurement, i.e. they use indices which compare prices and quantities of a group of commodities from the current period with the

[1]University of Lodz, Department of Statistical Methods, Lodz, Poland, jacek.bialek@uni.lodz.pl & Statistics Poland, Department of Trade and Services, Poland, J.Bialek@stat.gov.pl. ORCID: https://orcid.org/0000-0002-0952-5327.

corresponding prices and quantities from a base (fixed) period. A multilateral index is compiled over a given time window composed of $T+1$ successive months (typically $T = 12$). Multilateral price indices take as input all prices and quantities of the previously defined individual products, which are available in a given time window, i.e. in at least two of its periods. These methods are a very good choice in the case of dynamic scanner data, where we observe a large rotation of products and strong seasonality (Chessa et al., 2017). Moreover, multilateral indices are transitive, which means in practice that the calculation of the price dynamics for any two moments in the time window does not depend on the choice of the base period. By definition, transitivity eliminates the chain drift problem which may occur while using scanner data. The chain drift can be formalized in terms of the violation of the multi period identity test. According to this test, one can expect that when all prices and quantities in a current period revert back to their values from the base period, then the index should indicate no price change and it equals one. Thus, multilateral indices are free from the chain drift within a given estimation time window $[0,T]$. Although Ivancic et al. (2011) have suggested that the use of multilateral indices in the scanner data case can solve the chain drift problem, most statistical agencies using scanner data still make use of the monthly chained Jevons index (Chessa et al., 2017).

The Jevons (1865) index is an unweighted bilateral formula and it is used at the elementary aggregation level in the traditional data collection. As the scanner data provide information on consumption, it seems more appropriate to use weighted indices. Unfortunately, bilateral weighted formulas do not take into account all information from the time window, while the frequently chained weighted indices (even superlative) may generate chain drift bias (Chessa, 2015) and therefore do not reflect a reasonable price change over longer time intervals. For this reason, many countries have experimented with multilateral indices or even implemented them for the regular production of price indices (Krsinich (2014), Inklaar and Diewert (2016), Chessa et al. (2017); Chessa (2019), Diewert and Fox (2018), de Haan et al. (2021)).

Following the so-called identity test (International Labour Office, 2004; von der Lippe, 2007), however, one may expect that even when only prices return to their original values and quantities do not, the index becomes one. This test is quite restrictive for multilateral indices and causes some controversy among price statisticians. Nevertheless, it is mentioned among the axioms regarding multilateral indices both in the publications of the European Commission and in journals from the area of official statistics (Zhang et al., 2019). Unfortunately, the commonly used multilateral indices (GEKS, CCDI, GK, TPD, TDH) do not meet the identity test. The main aim of the paper is to present and discuss the proposition of two multilateral indices, the idea of which resembles the GEKS index, but which meet the identity test and most of other axioms. The proposed indices are compared with the multilateral SPQ index method, which is relatively new and also meets the identity test.

2. The list of considered multilateral price index methods

Multilateral index methods originate in comparisons of price levels across countries or regions. Commonly known methods include the GEKS method (Gini, 1931; Eltetö and Köves, 1964), the Geary-Khamis (GK) method (Geary, 1958; Khamis, 1972), the CCDI method (Caves et al., 1982) or the Time Product Dummy Methods (de Haan and Krsinich, 2018). These indices work on the defined time window $[0,T]$. The idea of the SPQ multilateral price index is based on the relative price and quantity dissimilarity measure Δ_{SPQ} (Diewert, 2020). The price dissimilarity measure is used to link together the bilateral Fisher indices according to the special algorithm, which extends the considered time window in each step.

Before we present the proposed multilateral price indices, let us denote sets of homogeneous products belonging to the same product group in months 0 and t by G_0 and G_t respectively, and let $G_{0,t}$ denote a set of matched products in both moments 0 and t. Although, in general, the item universe may be very dynamic in the scanner data case, we assume that there exits at least one product being available during the whole time interval $[0,T]$. Let p_i^τ and q_i^τ denote the price and quantity of the i-th product at time τ and $N_{0,t} = \operatorname{card} G_{0,t}$.

Since the indices proposed in the work are based on the idea of the GEKS index, let us recall its structure (see Section 2.1).

2.1. The GEKS method

Let us consider a time interval $[0,T]$ of observations of prices and quantities that will be used for constructing the GEKS index. The GEKS price index between months 0 and t is an unweighted geometric mean of $T+1$ ratios of bilateral price indices $P^{\tau,t}$ and $P^{\tau,0}$, which are based on the same price index formula. The bilateral price index formula should satisfy the time reversal test, i.e. it should satisfy the condition $P^{a,b} \cdot P^{b,a} = 1$. Typically, the GEKS method uses the superlative Fisher (1922) price index, resulting in the following formula:

$$P_{GEKS}^{0,t} = \prod_{\tau=0}^{T} \left(P_F^{0,\tau} P_F^{\tau,t} \right)^{\frac{1}{T+1}} . \tag{1}$$

Please note that de Haan and van der Grient (2011) suggested that the Törnqvist price index formula (Törnqvist, 1936) could be used instead of the Fisher price index in the Gini methodology. Following Diewert and Fox (2018), the multilateral price comparison method involving the GEKS method based on the Törnqvist price index is called the CCDI method.

3. Axiomatic approach in the multilateral method selection

According to the axiomatic approach, desirable index properties (the so-called "tests") are defined that a multilateral index may, or may not satisfy. The list of tests for multilateral indices can be found in the guide provided by the Australian Bureau of Statistics (2016) (see the chapter entitled: "CRITERIA FOR ASSESSING MULTILATERAL METHODS"). Interesting considerations concerning tests for price indices in the case of dynamic scanner data sets can be found in Zhang et al. (2019), where the authors - on the basis of the COLI (Cost of Living Index) and COGI (Cost of Goods Index) concepts - focus on five main test for a dynamic item universe (*identity test, fixed basket test, upper bound test, lower bound test* and *responsiveness test*).

Following the guidelines from the Australian Bureau of Statistics (2016) or the paper by Diewert (2020), we consider a wide set of tests for multilateral indices (see **Appendix A**) assuming that the conditions for their use are met (e.g. a set of matched products over a period of time is never empty).

Please note that the discussed multilateral index formulas (GK, GEKS, CCDI, TPD) meet most of the requirements at the same time, such as the *transitivity, multi-period identity test, positivity and continuity, proportionality, homogeneity in prices, commensurability, symmetry in the treatment of time periods* or *symmetry in treatment of products* tests. However, the discussed indexes differ in terms of the total set of tests they meet. For instance: the GEKS, CCDI and TPD indices do not satisfy the *basket test*, the Geary-Khamis and TPD indices do not satisfy the *responsiveness test to imputed prices* while the GEKS or CCDI can incorporate the imputed prices of missing products, and the *homogeneity in quantities* does not hold in the case of the Geary-Khamis formula. Please also note that the SPQ index is the only multilateral index that satisfies the *identity test*, which is a stronger requirement than the lack of chain drift.

4. Proposition of new multilateral indices

In the "classical" approach to constructing the GEKS-type indices, the bilateral price index formula, which is used in the GEKS' body, is the superlative one. In other words, although the standard GEKS method uses the Fisher indices as inputs (Chessa et al., 2017), other superlative indices are possible choices as well, e.g. the Törnqvist or Walsh indices (van Loon and Roels, 2018; Diewert and Fox, 2018). Moreover, in the paper by Chessa et al. (2017), we can read that "the bilateral indices should satisfy the time reversal test". The choice of the superlative indices as an input for GEKS has its justification in the economic approach, since the superlative indices are considered as to be the best proxies for the Cost of Living Index (International Labour Office, 2004). Please note, however, that the concept of multilateral indices is not based on the COLI framework and requirements for multilateral methods differ from those dedicated to bilateral ones. The *time reversibility* requirement, which allows the GEKS index to be transitive, enables expressing the GEKS index in a more intuitive, quotient form:

$$P_{GEKS}^{0,t} = \prod_{\tau=0}^{T} \left(\frac{P^{\tau,t}}{P^{\tau,0}} \right)^{\frac{1}{T+1}}. \tag{2}$$

where $P^{\tau,s}$ is the chosen bilateral price index formula (for $s = 0, t$).

In the next part of the work, two new multilateral indices, the structure of which may resemble the idea of the GEKS index at first glance, were proposed. However, the structure of the base index of the proposed multilateral formulas differs completely from the adopted convention related to the application of the superlative index. Moreover, the calculation of the base index will require quality adjusting, which in turn is more like the Geary-Khamis index idea. In fact, the proposed indices are in a sense a hybrid approach, i.e. they constitute a bridge between the quality adjusted unit value method and the GEKS method.

Finally, it should also be emphasised that one of the proposals (i.e. GEKS-AQU, see

Section 4.1) does not assume that the formula $P^{\tau,s}$ is a price index, but only a variant of quality adjusted unit value. This is a completely new approach in the theory of multilateral indices but still guarantees good axiomatic properties of the proposed index.

4.1. Proposition based on the asynchronous quality-adjusted unit value

In the unit value concept, prices of homogeneous products are equal to the ratio of expenditure and quantity sold (International Labour Office, 2004; Chessa et al., 2017). However, quantities of different products cannot be added together as in the case of homogeneous products. That is why the idea of quality-adjusted unit value assumes that prices p_i^s of different products $i \in G_s$ in month s are transformed into "quality-adjusted prices" $\frac{p_i^s}{v_i}$ and quantities q_i^s are converted into "common units" $v_i q_i^s$ by using a set of factors $v = \{v_i : i \in G_s\}$ (Chessa et al., 2017). Thus, the "classical" quality adjusted unit value $QUV_{G_s}^s$ of a set of products G_s in month s can be expressed as follows:

$$QUV_{G_s}^s = \frac{\sum_{i \in G_s} q_i^s p_i^s}{\sum_{i \in G_s} v_i q_i^s} \tag{3}$$

The term “Quality-adjusted unit value method” (QU method for short) was introduced by Chessa (2015; 2016). The QU method is a family of unit value based index methods and its general form can be expressed by the following ratio:

$$P_{QU}^{0,t} = \frac{QUV_{G_t}^t}{QUV_{G_0}^0} \tag{4}$$

In practice, consumer response to price changes can be delayed or even accelerated as consumers not only react to current price changes but also use their own "forecasts" or concerns about future price increases. For example, consumption of thermophilic (seasonal) fruit is likely to be higher in summer because they are cheaper than in winter, when the season is almost over. For instance, some interesting study on "unconventional" consumer behaviour, such as stocking and delayed quantity responses to price changes, and its impact on chain drift bias can be found in the paper by von Auer (2019). Since in practice we often observe prices and quantities that are not perfectly synchronised in time, the following form of the "asynchronous quality-adjusted unit value" is proposed:

$$AQUV_{G_{\tau,s}}^{\tau,s} = \frac{\sum_{i \in G_{\tau,s}} q_i^\tau p_i^s}{\sum_{i \in G_{\tau,s}} v_i q_i^\tau}, \tag{5}$$

where τ is any period from the considered time interval $[0,T]$. Obviously it holds that $AQUV_{G_{s,s}}^{s,s} = QUV_{G_s}^s$. Let us define now the function $P^{\tau,s}(v, q^\tau, p^\tau, p^s)$ as follows:

$$P^{\tau,s}(v, q^\tau, p^\tau, p^s) = \frac{AQUV_{G_{\tau,s}}^{\tau,s}}{AQUV_{G_{\tau,\tau}}^{\tau,\tau}}. \tag{6}$$

Putting (6) in formula (2) we obtain:

$$P_{GEKS-AQU}^{0,t} = \prod_{\tau=0}^{T} \left(\frac{\frac{\sum_{i \in G_{\tau,t}} q_i^\tau p_i^t}{\sum_{i \in G_{\tau,t}} v_i q_i^\tau}}{\frac{\sum_{i \in G_{\tau,0}} q_i^\tau p_i^0}{\sum_{i \in G_{\tau,0}} v_i q_i^\tau}} \right)^{\frac{1}{T+1}}. \tag{7}$$

Please note that the proposed index behaves like a GEKS index based on the Laspeyres index in the case of static item universe G. In fact, if the item universe is static, we obtain

$$\begin{aligned} P_{GEKS-AQU}^{0,t} &= \prod_{\tau=0}^{T} \left(\frac{\frac{\sum_{i \in G} q_i^\tau p_i^t}{\sum_{i \in G} v_i q_i^\tau}}{\frac{\sum_{i \in G} q_i^\tau p_i^0}{\sum_{i \in G} v_i q_i^\tau}} \right)^{\frac{1}{T+1}} = \prod_{\tau=0}^{T} \left(\frac{\sum_{i \in G} q_i^\tau p_i^t}{\sum_{i \in G} q_i^\tau p_i^0} \right)^{\frac{1}{T+1}} \\ &= \prod_{\tau=0}^{T} \left(\frac{\frac{\sum_{i \in G} q_i^\tau p_i^t}{\sum_{i \in G} q_i^\tau p_i^\tau}}{\frac{\sum_{i \in G} q_i^\tau p_i^0}{\sum_{i \in G} q_i^\tau p_i^\tau}} \right)^{\frac{1}{T+1}} = P_{GEKS-L}^{0,t}. \end{aligned} \tag{8}$$

Finally, please also note, that theoretically the class of the $GEKS-AQU$ indices is infinite, since different choices of v_i factors lead to different index values. We could, for instance, consider v_i factors defined in the Geary-Khamis multilateral index resulting in a new, hybrid index, which would be a mixture of the GEKS and Geary-Khamis ideas. That would, however, be probably a slow solution. In this paper, we adopt the system of weights v_i corresponding to the augmented Lehr index (Lamboray, 2017; van Loon and Roels, 2018), where

$$v_i = \frac{\sum_{t=0}^{T} p_i^t q_i^t}{\sum_{t=0}^{T} q_i^t}. \tag{9}$$

The following theorem can be proved (see **Appendix B**):

Theorem 1 *The GEKS-AQU index (7) satisfies the following tests: the transitivity, identity, multi period identity, responsiveness, continuity, positivity and normalisation, price proportionality and weak commensurability. If the item universe is the same in the compared periods* 0 *and* t *then the GEKS-AQU index satisfies also the homogeneity in prices and homogeneity in quantities tests.*

4.2. Proposition based on the asynchronous quality-adjusted price index

Let us note that formula (5) can be expressed by using quality-adjusted prices and quantities:

$$AQUV_{G_{\tau,s}}^{\tau,s} = \frac{\sum_{i \in G_{\tau,s}} v_i q_i^\tau \frac{p_i^s}{v_i}}{\sum_{i \in G_{\tau,s}} v_i q_i^\tau}. \tag{10}$$

If we place all the adjusted prices ($\frac{p_i^s}{v_i}$) with the relative prices ($\frac{p_i^s}{p_i^\tau}$), then we obtain an "asynchronous quality-adjusted price index" (AQI), i.e.

$$AQI_{G_{\tau,s}}^{\tau,s} = \frac{\sum_{i \in G_{\tau,s}} v_i q_i^\tau \frac{p_i^s}{p_i^\tau}}{\sum_{i \in G_{\tau,s}} v_i q_i^\tau}. \tag{11}$$

This means that the AQI formula can be treated as a weighted arithmetic mean of partial indices $\frac{p_i^s}{p_i^\tau}$, where the weights are proportional to the relative share of the product's adjusted quantities (from the base period τ) in the sum of all adjusted quantities.

In the further part of the work, the GEKS index based on the AQI formula will be marked as GEKS-AQI, i.e. by inserting (11) into the formula (2), we obtain:

$$P_{GEKS-AQI}^{0,t} = \prod_{\tau=0}^{T} \left(\frac{\frac{\sum_{i \in G_{\tau,t}} v_i q_i^\tau \frac{p_i^t}{p_i^\tau}}{\sum_{i \in G_{\tau,t}} v_i q_i^\tau}}{\frac{\sum_{i \in G_{\tau,0}} v_i q_i^\tau \frac{p_i^0}{p_i^\tau}}{\sum_{i \in G_{\tau,0}} v_i q_i^\tau}} \right)^{\frac{1}{T+1}}. \tag{12}$$

Note that the GEKS-AQI index takes into account prices and quantities directly from all time window periods, while the GEKS-AQU index takes into account all quantities but only prices from the reference and base period. However, both formulas indirectly need information about the prices (and quantities) of products from each period in the time window to determine the factors v_i defined by formula (9). In this way, each new product in the analysed time window has an impact on the final value of the proposed indices.

It is possible to show, analogously to the proofs of Theorem 1 (see **Appendix B**), that the following theorem holds:

Theorem 2 *The GEKS-AQI index (12) satisfies the following tests: the transitivity, identity, multi period identity, responsiveness, continuity, positivity and normalisation, price proportionality and weak commensurability. If the item universe is the same in the compared periods* 0 *and* t *then the GEKS-AQI index satisfies also the homogeneity in prices and homogeneity in quantities test.*

Remark

Similarly to the weighted GEKS index (Melser, 2018), it seems to be interesting to consider the following weighted versions of the GEKS-AQU and GEKS-AQI indices:

$$P_{WGEKS-AQU}^{0,t} = \prod_{\tau=0}^{T} \left(\frac{\frac{\sum_{i \in G_{\tau,t}} q_i^\tau p_i^t}{\sum_{i \in G_{\tau,t}} v_i q_i^\tau}}{\frac{\sum_{i \in G_{\tau,0}} q_i^\tau p_i^0}{\sum_{i \in G_{\tau,0}} v_i q_i^\tau}} \right)^{v_\tau}, \tag{13}$$

and

$$P_{WGEKS-AQI}^{0,t} = \prod_{\tau=0}^{T} \left(\frac{\frac{\sum_{i \in G_{\tau,t}} v_i q_i^{\tau} \frac{p_i^t}{p_i^{\tau}}}{\sum_{i \in G_{\tau,t}} v_i q_i^{\tau}}}{\frac{\sum_{i \in G_{\tau,0}} v_i q_i^{\tau} \frac{p_i^0}{p_i^{\tau}}}{\sum_{i \in G_{\tau,0}} v_i q_i^{\tau}}} \right)^{v_\tau}, \tag{14}$$

where the weights concerning the period τ can be, for instance, defined as follows:

$$v_\tau = \frac{\sum_{i \in G_\tau} q_i^{\tau} p_i^{\tau}}{\sum_{\tau=0}^{T} \sum_{i \in G_\tau} q_i^{\tau} p_i^{\tau}}. \tag{15}$$

5. Empirical Study

Scanner data from one retail chain in Poland are used in our empirical study, i.e. monthly data on *long grain rice* (subgroup of COICOP 5 group: 011111), *ground coffee* (subgroup of COICOP 5 group: 012111), *drinking yoghurt* (subgroup of COICOP 5 group: 011441) and *white sugar* (subgroup of COICOP 5 group: 011811) sold in 212 outlets during the period from December 2019 to December 2020 (352705 records, which means 210 MB of data). Before price index calculations, the data sets were carefully prepared. First, after deleting the records with missing data and performing the deduplication process, the products were classified into the relevant elementary groups (COICOP 5 level) and, after that, into their subgroups (local COICOP 6 level). The classification process was performed using the `data_selecting()` and `data_classification()` functions from the `PriceIndices` R package (Białek, 2021). The first function requires manual preparation of dictionaries of keywords and phrases that identified individual product groups. The second function was used for problematic, previously unclassified products, and required manual preparation of learning samples based on historical data. The classification itself was based on machine learning techniques using random trees and the XGBoost algorithm (Tianqi and Carlo, 2016). To match products, we used the `data_matching()` function from the `PriceIndices` package. To be more precise: products with two identical codes or one of the codes identical and an identical description were automatically matched. Products were also matched if they had identical one of the codes and the Jaro-Winkler (1989) distance of their descriptions was smaller than the fixed precision value: 0.02. In the last step, just before calculating price indices, two data filters were applied to remove unrepresentative products from the database, i.e. the `data_filtering()` function from the cited package was used. The *extreme price filter* (Białek and Beręsewicz, 2021) was applied to eliminate products with more than a three-fold price increase or more than a double price drop from month to month. The *low sale filter* (van Loon and Roels, 2018) was used to eliminate from the sample products with relatively low sales (almost 30% of products were removed).

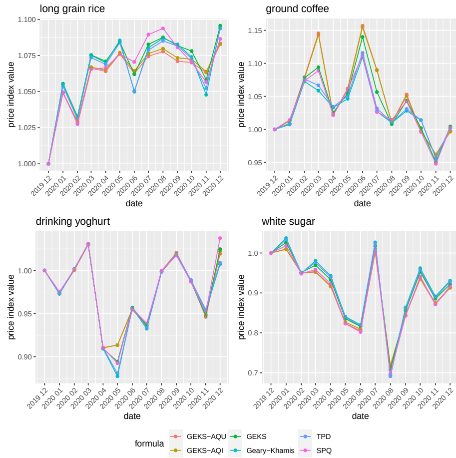

Figure 1: Comparison of selected multilateral indices for four homogeneous groups of food products

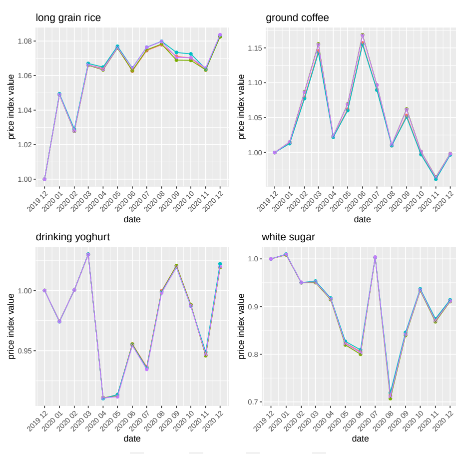

Figure 2: Comparison of the GEKS-AQU and GEKS-AQI indices to their weighted versions for four homogeneous groups of food products

Table 1: Mean absolute differences between considered price indices calculated for **long grain rice**: Dec, 2019 - Dec, 2020 (p.p.)

index	GEKS-AQU	GEKS-AQI	WGEKS-AQU	WGEKS-AQI	GEKS	GK	TPD	SPQ
GEKS-AQU	0.00	0.12	0.06	0.07	0.75	0.87	0.71	0.53
GEKS-AQI	0.12	0.00	0.16	0.08	0.67	0.78	0.63	0.49
WGEKS-AQU	0.06	0.16	0.00	0.09	0.80	0.91	0.76	0.55
WGEKS-AQI	0.07	0.08	0.09	0.00	0.75	0.86	0.71	0.49
GEKS	0.75	0.67	0.80	0.75	0.00	0.29	0.33	0.62
GK	0.87	0.78	0.91	0.86	0.29	0.00	0.15	0.76
TPD	0.71	0.63	0.76	0.71	0.33	0.15	0.00	0.66
SPQ	0.53	0.49	0.55	0.49	0.62	0.76	0.66	0.00

Table 2: Mean absolute differences between considered price indices calculated for **ground coffee**: Dec, 2019 - Dec, 2020 (p.p.)

index	GEKS-AQU	GEKS-AQI	WGEKS-AQU	WGEKS-AQI	GEKS	GK	TPD	SPQ
GEKS-AQU	0.00	0.11	0.54	0.50	1.27	2.39	2.16	1.75
GEKS-AQI	0.11	0.00	0.65	0.62	1.19	2.30	2.07	1.66
WGEKS-AQU	0.54	0.65	0.00	0.04	1.71	2.83	2.58	2.24
WGEKS-AQI	0.50	0.62	0.04	0.00	1.68	2.80	2.56	2.20
GEKS	1.27	1.19	1.71	1.68	0.00	1.30	1.06	0.83
GK	2.39	2.30	2.83	2.80	1.30	0.00	0.27	0.94
TPD	2.16	2.07	2.58	2.56	1.06	0.27	0.00	0.87
SPQ	1.75	1.66	2.24	2.20	0.83	0.94	0.87	0.00

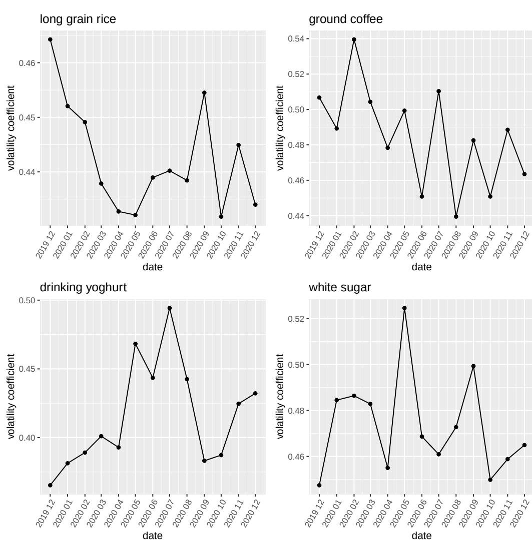

Figure 3: Monthly coefficients of variation of prices calculated for the studied four product groups

Table 3: Mean absolute differences between considered price indices calculated for **drinking yoghurt**: Dec, 2019 - Dec, 2020 (p.p.)

index	GEKS-AQU	GEKS-AQI	WGEKS-AQU	WGEKS-AQI	GEKS	GK	TPD	SPQ
GEKS-AQU	0.00	0.09	0.04	0.08	0.30	0.56	0.50	0.44
GEKS-AQI	0.09	0.00	0.10	0.08	0.25	0.55	0.49	0.40
WGEKS-AQU	0.04	0.10	0.00	0.06	0.30	0.55	0.49	0.45
WGEKS-AQI	0.08	0.08	0.06	0.00	0.28	0.54	0.48	0.44
GEKS	0.30	0.25	0.30	0.28	0.00	0.39	0.36	0.22
GK	0.56	0.55	0.55	0.54	0.39	0.00	0.08	0.52
TPD	0.50	0.49	0.49	0.48	0.36	0.08	0.00	0.47
SPQ	0.44	0.40	0.45	0.44	0.22	0.52	0.47	0.00

Table 4: Mean absolute differences between considered price indices calculated for **white sugar**: Dec, 2019 - Dec, 2020 (p.p.)

index	GEKS-AQU	GEKS-AQI	WGEKS-AQU	WGEKS-AQI	GEKS	GK	TPD	SPQ
GEKS-AQU	0.00	0.21	0.26	0.10	1.24	2.04	1.79	0.50
GEKS-AQI	0.21	0.00	0.47	0.30	1.11	1.92	1.67	0.59
WGEKS-AQU	0.26	0.47	0.00	0.18	1.39	2.19	1.93	0.58
WGEKS-AQI	0.10	0.30	0.18	0.00	1.27	2.08	1.83	0.49
GEKS	1.24	1.11	1.39	1.27	0.00	0.82	0.56	0.99
GK	2.04	1.92	2.19	2.08	0.82	0.00	0.28	1.61
TPD	1.79	1.67	1.93	1.83	0.56	0.28	0.00	1.38
SPQ	0.50	0.59	0.58	0.49	0.99	1.61	1.38	0.00

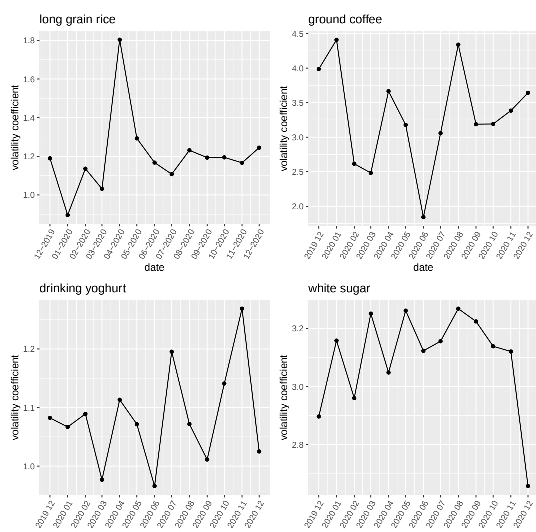

Figure 4: Monthly coefficients of variation of quantities calculated for the studied four product groups

The results obtained for the GEKS-AQU, GEKS-AQI, GEKS, Geary-Khamis, TPD and SPQ indices are presented in Figure 1. An additional comparison of the GEKS-AQU and GEKS-AQI indices to their weighted versions is presented in Figure 2. The average absolute differences between the indices were determined by using the `compare_distances()` function from the `PriceIndices` package (see Tab.1 - Tab.4). Observing Fig.2 and Tab.1-4 we conclude that the largest differences between the GEKS-AQU and GEKS-AQI indices and their weighted versions were observed for data sets: ground coffee (average absolute differences of about 0.5 p.p.) and white sugar (average absolute differences between 0.1 and 0.26 p.p.). In the case of the other two data sets, the corresponding differences turned out to be neglectfully small (they do not exceed 0.09 p.p.). Similarly, the differences between the other multilateral indices appear to be the largest for these ground coffee and white sugar data sets (see Fig.1). As a consequence, there is a suspicion that the distribution of the values of weights based on expenditure shares determines the differences between multilateral indices. However, since the prices of products from homogeneous groups are unlikely to be as diverse as the possible sales levels of these products, there is a natural research hypothesis that the volatility of quantities is the main cause of the differences between multilateral indices. To verify the above-presented hypothesis, an additional analysis was made by determining the monthly coefficients of prices and quantities for all product groups (see Fig.3 and Fig.4).

Price volatility (measured by the coefficient of variation), which is the main cause of differences between bilateral price indices, turned out not to differentiate the analyzed data sets (see Fig.3), and thus it was not price volatility that determined the differences between the values of the indices. As previously suspected, the volatility of the quantity of products sold seems to have a clear impact on the differences between multilateral indices. This conclusion confirms previously obtained results (Białek, 2022). Please note that the coefficients of variation of product quantities are clearly higher for the data sets for white sugar and ground coffee (Fig.4). However, this thread requires further research.

As it was mentioned above, the GEKS-AQU and GEKS-AQI indices approximate each other. Moreover, their values are quite close to those of the GEKS and SPQ indices. The Geary-Khamis index is a good proxy for the Time Product Dummy (TPD) index, which confirms some previous results (Chessa et al., 2017; Białek and Beręsewicz, 2021), but it always seems to be the most distant from the proposed indices.

In terms of how time-consuming their calculations are, the proposed indices seem to be average. More precisely: the GEKS-AQU index requires slightly less computing time than the GEKS-AQI index - in this respect it is better than the TPD or Geary-Khamis indices, but is worse (slower in calculation) than the SPQ or GEKS indices. The last conclusion, however, is not surprising: firstly, the SPQ index does not work on a given time window like other multilateral indices, and secondly, the GEKS index does not perform quality adjustment as the GEKS-AQU and GEKS-AQI indices do.

6. Conclusions

The paper proposes two new multilateral indices, the idea of which resembles the GEKS method, but which perform additional quality adjustment and deviate from the classical approach in which the base formula of the GEKS index is a superlative index. The analytical study has confirmed that the two proposed indices (GEKS-AQU and GEKS-AQI) satisfy most of commonly accepted tests for multilateral indices (see Appendix 6) including the *identity test* (see Theorems 1 and Theorem 2). The empirical study has shown that differences between the proposed indices and other considered multilateral indices appear only with large variability of quantity in homogeneous groups of products. Quite surprisingly, the price volatility did not play a significant role in the empirical study as determinants of differences between multilateral indices (see Section 5). The same study has also shown that the computation time needed in the case of the GEKS-AQU and GEKS-AQI indices is average compared to most other multilateral indices.

It should also be noted that both the previously known multilateral indices (Geary-Khamis, GEKS, TPD, CCDI, and SPQ) as well as the new indices proposed and discussed in the paper (GEK-AQU, GEKS-AQI, and their weighted versions: WGEKS-AQU and WGEKS-AQI) are implemented in the PriceIndices R package (Białek, 2021), and thus the reader can verify their usefulness on their own data sets.

Nevertheless, some aspects of the behaviour of the proposed multilateral indices still remain unexplored. From a practical point of view, it seems interesting, how great the sensitivity of these methods to changing window updating methods is or if the selection of filter thresholds has a considerable influence on the index values. It should be also noted that the issue of choosing between weighted and unweighted versions of a GEKS-type index is not resolved in the literature. On the one hand, the GEKS-type method uses an underlying, bilateral price index that already performs the first weighting. Thus, it seems that additional weighting is an unnecessary waste of time and may even be detrimental due to giving too much weight to leading products and too little weight to products with relatively lower sales. On the other hand, however, the second weighting stage ranks all periods from the time window, whereas the first weighting stage only looks at pairs of periods being compared. Such a dual weighting system can therefore be an alternative to the low sales filter, i.e. it can be considered when one does not choose to pre-filter the data set. From a purely axiomatic point of view, however, it should be borne in mind that unweighted versions of the GEKS index are well recognized in the literature, meanwhile it is uncertain whether the introduction of additional weighting will not affect the set of tests (axioms) satisfied by a given formula.

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References

Australian Bureau of Statistics (2016), *Making Greater Use of Transactions Data to Compile the Consumer Price Index*, Information Paper 6401.0.60.003, Canberra.

Białek, J. (2021), 'Priceindices – a new R package for bilateral and multilateral price index calculations', *Statistika – Statistics and Economy Journal* **36**(2), 122–141.

Białek, J. and Beręsewicz, M. (2021), 'Scanner data in inflation measurement: from raw data to price indices', *The Statistical Journal of the IAOS* **37**, 1315–1336.

Białek, J. (2022), 'Improving quality of the scanner CPI: proposition of new multilateral methods', *Quality and Quantity* **In press, online at: https://doi.org/10.1007/s11135-022-01506-6**.

Caves, D. W., Christensen, L. R. and Diewert, W. E. (1982), 'Multilateral comparisons of output, input, and productivity using superlative index numbers', *Economic Journal* **92**(365), 73–86.

Chessa, A. (2015), Towards a generic price index method for scanner data in the dutch cpi, *in* '14th meeting of the Ottawa Group, Tokyo', pp. 20–22.

Chessa, A. (2016), 'A new methodology for processing scanner data in the Dutch CPI', *Eurostat review of National Accounts and Macroeconomic Indicators* **1**, 49–69.

Chessa, A. (2019), A comparison of index extension methods for multilateral methods, *in* 'Paper presented at the 16th Meeting of the Ottawa Group on Price Indices, Rio de Janeiro, Brazil'.

Chessa, A. G., Verburg, J. and Willenborg, L. (2017), A comparison of price index methods for scanner data, *in* '15th meeting of the Ottawa Group, Eltville', pp. 10–12.

de Haan, J., Hendriks, R. and Scholz, M. (2021), 'Price measurement using scanner data: Time-product dummy versus time dummy hedonic indexes', *Review of Income and Wealth* **67**(2), 394–417.

de Haan, J. and Krsinich, F. (2018), 'Time dummy hedonic and quality-adjusted unit value indexes: Do they really differ?', *Review of Income and Wealth* **64**(4), 757–776.

de Haan, J. and van der Grient, H. A. (2011), 'Eliminating chain drift in price indexes based on scanner data', *Journal of Econometrics* **161**(1), 36–46.

Diewert, W. E. (2020), The chain drift problem and multilateral indexes, Technical report, Discussion Paper 20-07, Vancouver School of Economics.

Diewert, W. E. and Fox, K. J. (2018), 'Substitution bias in multilateral methods for CPI construction using scanner data', *UNSW Business School Research Paper* (2018-13).

Eltetö, O. and Köves, P. (1964), 'On a problem of index number computation relating to international comparison', *Statisztikai Szemle* **42**(10), 507–518.

Fisher, I. (1922), *The making of index numbers: a study of their varieties, tests, and reliability*, Vol. xxxi, Houghton Mifflin.

Geary, R. C. (1958), 'A note on the comparison of exchange rates and purchasing power between countries', *Journal of the Royal Statistical Society. Series A (General)* **121**(1), 97–99.

Gini, C. (1931), 'On the circular test of index numbers', *Metron* **9**(9), 3–24.

Inklaar, R. and Diewert, W. E. (2016), 'Measuring industry productivity and cross-country convergence', *Journal of Econometrics* **191**(2), 426–433.

International Labour Office (2004), 'Consumer Price Index Manual: Theory and Practice', Geneva.

Ivancic, L., Diewert, W. E. and Fox, K. J. (2011), 'Scanner data, time aggregation and the construction of price indexes', *Journal of Econometrics* **161**(1), 24–35.

Jaro, M. (1989), 'Advances in record-linkage methodology as applied to matching the 1985 census of Tampa, Florida', *Journal of the American Statistical Association* **84**(406), 414–420.

Jevons, W. S. (1865), 'On the variation of prices and the value of the currency since 1782', *Journal of the Statistical Society of London* **28**(2), 294–320.

Khamis, S. H. (1972), 'A new system of index numbers for national and international purposes', *Journal of the Royal Statistical Society: Series A (General)* **135**(1), 96–121.

Krsinich, F. (2014), The FEWS Index: Fixed Effects with a Window Splice–Non-Revisable Quality-Adjusted Price Indexes with No Characteristic Information, *in* 'Meeting of the group of experts on consumer price indices', pp. 26–28.

Lamboray, C. (2017), The Geary-Khamis index and the Lehr index: how much do they differ, *in* 'Paper to be presented at the 15th meeting of the Ottawa Group', pp. 10–12.

Melser, D. (2018), 'Scanner data price indexes: Addressing some unresolved issues', *Journal of Business and Economic Statistics* **36**(03), 516–522.

Tianqi, C. and Carlo, G. (2016), Xgboost: A scalable tree boosting system, *in* 'Proceedings of the 22nd ACM SIGKDD International Conference on Knowledge Discovery and Data Mining, San Francisco, CA, USA, August 13-17, 2016. ACM', p. 785–794.

Törnqvist, L. (1936), 'The bank of Finland's consumption price index', *Bank of Finland Monthly Bulletin* pp. 1–8.

van Loon, K. V. and Roels, D. (2018), Integrating big data in the Belgian CPI, *in* 'Paper presented at the meeting of the group of experts on consumer price indices, 8-9 May 2018, Geneva, Switzerland'.

von Auer, L. (2019), The nature of chain drift, *in* 'Paper presented at the 17th Meeting of the Ottawa Group on Price Indices, 8–10 May 2019, Rio de Janerio, Brasil'.

von der Lippe, P. (2007), *Index Theory and Price Statistics*, Peter Lang, Berlin, Germany.

Zhang, L.-C., Johansen, I. and Nyagaard, R. (2019), 'Tests for price indices in a dynamic item universe', *Journal of Official Statistics* **35**(3), 683–697.

Appendices

Appendix A: Tests for multilateral indices

Let P and Q denote all prices and quantities observed in the time interval $[0,T]$, i.e. $P=[p^0,p^1,...,p^T]$, $Q=[q^0,q^1,...,q^T]$, where p^t and q^t mean the vector of prices and the vector of quantities of products sold at time t, respectively. Let us denote by $P^{0,t}(P,Q)$ the considered multilateral price index defined for the entire time window $[0,T]$. The list of commonly accepted tests for that index is as follows:

Transitivity

The transitivity means that $P^{0,t}(P,Q)=P^{0,s}(P,Q)P^{s,t}(P,Q)$ for any $0\leq s<t\leq T$.

Identity

This property means that the index equals identity if all prices revert back to their initial level, i.e. if it holds that $p_i^t=p_i^0$ for $i\in G_{0,t}$ then $P^{0,t}(P,Q)=1$. We assume here that the item universe is the same at periods 0 and t.

Multi period identity test

This property means that if all prices and quantities revert back to their initial level, the chained index will equal the unity, i.e. if it holds that $p_i^t=p_i^0$ and $q_i^t=q_i^0$ for $i\in G_{0,t}$ then we obtain $P^{0,1}(P,Q)\times P^{1,2}(P,Q)\times....\times P^{t-1,t}(P,Q)=1$. We assume here that the item universe is the same at periods 0 and t.

Fixed basket test

If $G_0=G_t$ and $q_i^0=q_i^t=q_i$ for $i\in G_{0,t}$, then $P^{0,t}(P,Q)=\frac{\sum_{i\in G_{0,t}}p_i^tq_i}{\sum_{i\in G_{0,t}}p_i^0q_i}$.

Responsiveness test

For $G_0\neq G_t$, if $p_i^t=p_i^0$ for all $i\in G_{0,t}$, then $P^{0,t}(P,Q)$ cannot always equal one, regardless of sets: $G_0\setminus G_t$ and $G_t\setminus G_0$.

Continuity, positivity and normalization

$P^{0,t}(P,Q)$ is a positive and continuous function of prices and quantities, $P^{0,0}(P,Q)=1$.

Price proportionality

If all prices are proportional in the compared periods 0 and t, i.e. $p_i^t=kp_i^0$ for all $i\in G_{0,t}$ and some positive k, then the price index depends only on this proportion: $P^{0,t}(P,Q)=k$. We assume here that the item universe is the same at periods 0 and t.

Homogeneity in quantities

Rescaling the quantities in any s-th period does not influence the price index, i.e. for any positive k, it holds that $P^{0,t}(P,q^0,...,kq^s,...,q^t)=P^{0,t}(P,q^0,...,q^s,...,q^t)$.

Homogeneity in prices

Rescaling the prices in the current period changes the price index by the same proportion, i.e. for any positive k, it holds that $P^{0,t}(p^0, p^1, ..., kp^t, Q) = kP^{0,t}(p^0, p^1, ..., p^t, Q)$.

Commensurability

Changing the units in which prices and quantities are expressed does not change the price index. In other words, if for each time moment $s \in [0,T]$ we have $\tilde{p}_i^s = \lambda_i p_i^s$ and $\tilde{q}_i^s = \frac{q_i^s}{\lambda_i}$ for all $i \in G_s$ ($\lambda_i > 0$), then $P^{0,t}(\tilde{P}, \tilde{Q}) = P^{0,t}(P, Q)$. If this conditions holds but only for identical values of λ_i, i.e. when $\lambda_1 = \lambda_2 = ...\lambda_N = \lambda$, then the *weak commensurability* is satisfied.

Appendix B: Proof of Theorem 1

Transitivity

Let us consider such periods s and t from the time window $[0,T]$ that $0 \leq s < t \leq T$. We obtain

$$P^{0,s}_{GEKS-AQU} \times P^{s,t}_{GEKS-AQU} = \prod_{\tau=0}^{T}\left(\frac{\frac{\sum_{i\in G_{\tau,s}} q_i^\tau p_i^s}{\sum_{i\in G_{\tau,s}} v_i q_i^\tau}}{\frac{\sum_{i\in G_{\tau,0}} q_i^\tau p_i^0}{\sum_{i\in G_{\tau,0}} v_i q_i^\tau}}\right)^{\frac{1}{T+1}} \times \prod_{\tau=0}^{T}\left(\frac{\frac{\sum_{i\in G_{\tau,t}} q_i^\tau p_i^t}{\sum_{i\in G_{\tau,t}} v_i q_i^\tau}}{\frac{\sum_{i\in G_{\tau,s}} q_i^\tau p_i^s}{\sum_{i\in G_{\tau,s}} v_i q_i^\tau}}\right)^{\frac{1}{T+1}} =$$

$$= \prod_{\tau=0}^{T}\left(\frac{\frac{\sum_{i\in G_{\tau,t}} q_i^\tau p_i^t}{\sum_{i\in G_{\tau,t}} v_i q_i^\tau}}{\frac{\sum_{i\in G_{\tau,0}} q_i^\tau p_i^0}{\sum_{i\in G_{\tau,0}} v_i q_i^\tau}}\right)^{\frac{1}{T+1}} = P^{0,t}_{GEKS-AQU}$$

Identity

Let us assume that $G_0 = G_t = G_{0,t}$ and $p_i^t = p_i^0$ for $i \in G_{0,t}$. We have

$$P^{0,t}_{GEKS-AQU} = \prod_{\tau=0}^{T}\left(\frac{\frac{\sum_{i\in G_{\tau,t}} q_i^\tau p_i^t}{\sum_{i\in G_{\tau,t}} v_i q_i^\tau}}{\frac{\sum_{i\in G_{\tau,0}} q_i^\tau p_i^0}{\sum_{i\in G_{\tau,0}} v_i q_i^\tau}}\right)^{\frac{1}{T+1}} = \prod_{\tau=0}^{T}\left(\frac{\frac{\sum_{i\in G_{\tau,0}} q_i^\tau p_i^0}{\sum_{i\in G_{\tau,0}} v_i q_i^\tau}}{\frac{\sum_{i\in G_{\tau,0}} q_i^\tau p_i^0}{\sum_{i\in G_{\tau,0}} v_i q_i^\tau}}\right)^{\frac{1}{T+1}} = 1.$$

Multi period identity test

Let us assume that that $p_i^t = p_i^0$ and $q_i^t = q_i^0$ for $i \in G_{0,t} = G_0 = G_t$. We obtain

$$P^{0,1}_{GEKS-AQU} \times P^{1,2}_{GEKS-AQU} \times ... \times P^{t-1,t}_{GEKS-AQU} =$$

$$= \prod_{\tau=0}^{T}\left(\frac{\frac{\sum_{i\in G_{\tau,1}} q_i^\tau p_i^1}{\sum_{i\in G_{\tau,1}} v_i q_i^\tau}}{\frac{\sum_{i\in G_{\tau,0}} q_i^\tau p_i^0}{\sum_{i\in G_{\tau,0}} v_i q_i^\tau}}\right)^{\frac{1}{T+1}} \times \prod_{\tau=0}^{T}\left(\frac{\frac{\sum_{i\in G_{\tau,2}} q_i^\tau p_i^2}{\sum_{i\in G_{\tau,2}} v_i q_i^\tau}}{\frac{\sum_{i\in G_{\tau,1}} q_i^\tau p_i^1}{\sum_{i\in G_{\tau,1}} v_i q_i^\tau}}\right)^{\frac{1}{T+1}} \times ... \times \prod_{\tau=0}^{T}\left(\frac{\frac{\sum_{i\in G_{\tau,t}} q_i^\tau p_i^t}{\sum_{i\in G_{\tau,t}} v_i q_i^\tau}}{\frac{\sum_{i\in G_{\tau,t-1}} q_i^\tau p_i^{t-1}}{\sum_{i\in G_{\tau,t-1}} v_i q_i^\tau}}\right)^{\frac{1}{T+1}} =$$

$$= \prod_{\tau=0}^{T}\left(\frac{\frac{\sum_{i\in G_{\tau,t}} q_i^\tau p_i^t}{\sum_{i\in G_{\tau,t}} v_i q_i^\tau}}{\frac{\sum_{i\in G_{\tau,0}} q_i^\tau p_i^0}{\sum_{i\in G_{\tau,0}} v_i q_i^\tau}}\right)^{\frac{1}{T+1}} = \prod_{\tau=0}^{T}\left(\frac{\frac{\sum_{i\in G_{\tau,0}} q_i^\tau p_i^0}{\sum_{i\in G_{\tau,0}} v_i q_i^\tau}}{\frac{\sum_{i\in G_{\tau,0}} q_i^\tau p_i^0}{\sum_{i\in G_{\tau,0}} v_i q_i^\tau}}\right)^{\frac{1}{T+1}} = 1$$

Please note that this proof does not require the condition $q_i^t = q_i^0$.

Responsiveness test

Let us assume that $G_0 \neq G_t$ and $p_i^t = p_i^0$ for all $i \in G_{0,t}$. Since $G_0 \neq G_t$, we know that for at least one period τ_0 we have $G_{\tau_0,t} \neq G_{\tau_0,0} \cap G_{\tau_0,t}$, for at least one period τ_* we have $G_{\tau_*,0} \neq G_{\tau_*,0} \cap G_{\tau_*,t}$ and, from our initial assumption (see Section 2), we have that $G_{\tau,0} \cap G_{\tau,t} \neq \emptyset$ for any τ. As a consequence, in general we observe $AQUV_{G^*_{\tau_0,t}}^{\tau_0,t} \neq AQUV_{G_{\tau_0,t}}^{\tau_0,t}$, where $G^*_{\tau_0,t} = G_{\tau_0,0} \cap G_{\tau_0,t}$. In a similar way we can conclude that $AQUV_{G^*_{\tau_*,0}}^{\tau_*,0} \neq AQUV_{G_{\tau_*,0}}^{\tau_*,0}$, where $G^*_{\tau_*,0} = G_{\tau_*,0} \cap G_{\tau_*,t}$. Thus, from (6), it holds generally that

$$P_{GEKS-AQU}^{0,t} = \prod_{\tau=0}^{T}\left(\frac{\frac{\sum_{i\in G_{\tau,t}} q_i^\tau p_i^t}{\sum_{i\in G_{\tau,t}} v_i q_i^\tau}}{\frac{\sum_{i\in G_{\tau,0}} q_i^\tau p_i^0}{\sum_{i\in G_{\tau,0}} v_i q_i^\tau}}\right)^{\frac{1}{T+1}} \neq \prod_{\tau=0}^{T}\left(\frac{\frac{\sum_{i\in G_{\tau,0}\cap G_{\tau,t}} q_i^\tau p_i^t}{\sum_{i\in G_{\tau,0}\cap G_{\tau,t}} v_i q_i^\tau}}{\frac{\sum_{i\in G_{\tau,0}\cap G_{\tau,t}} q_i^\tau p_i^0}{\sum_{i\in G_{\tau,0}\cap G_{\tau,t}} v_i q_i^\tau}}\right)^{\frac{1}{T+1}}.$$

Since we assume that prices at compared time moments are identical, i.e. $p_i^t = p_i^0$ for $i \in G_{\tau,0} \cap G_{\tau,t}$, we obtain that

$$P_{GEKS-AQU}^{0,t} \neq \prod_{\tau=0}^{T}\left(\frac{\frac{\sum_{i\in G_{\tau,0}\cap G_{\tau,t}} q_i^\tau p_i^0}{\sum_{i\in G_{\tau,0}\cap G_{\tau,t}} v_i q_i^\tau}}{\frac{\sum_{i\in G_{\tau,0}\cap G_{\tau,t}} q_i^\tau p_i^0}{\sum_{i\in G_{\tau,0}\cap G_{\tau,t}} v_i q_i^\tau}}\right)^{\frac{1}{T+1}} = 1.$$

Continuity, positivity and normalisation

Continuity and *positivity* are direct consequences of the definition of the GEKS-AQU index. The *normalisation* property is also an immediate consequence from its form (6), i.e.

$$P_{GEKS-AQU}^{0,0} = \prod_{\tau=0}^{T}\left(\frac{\frac{\sum_{i\in G_{\tau,0}} q_i^\tau p_i^0}{\sum_{i\in G_{\tau,0}} v_i q_i^\tau}}{\frac{\sum_{i\in G_{\tau,0}} q_i^\tau p_i^0}{\sum_{i\in G_{\tau,0}} v_i q_i^\tau}}\right)^{\frac{1}{T+1}} = 1.$$

Price proportionality

Assumption that the item universe is the same at periods 0 and t means that $G_0 = G_t = G_{0,t}$ and also $G_{\tau,0} = G_{\tau,t}$ for any τ. Let us assume that $p_i^t = kp_i^0$ for all $i \in G_{0,t}$ and some positive k. As a consequence, we obtain

$$P_{GEKS-AQU}^{0,t} = \prod_{\tau=0}^{T}\left(\frac{\frac{\sum_{i\in G_{\tau,t}} q_i^\tau p_i^t}{\sum_{i\in G_{\tau,t}} v_i q_i^\tau}}{\frac{\sum_{i\in G_{\tau,0}} q_i^\tau p_i^0}{\sum_{i\in G_{\tau,0}} v_i q_i^\tau}}\right)^{\frac{1}{T+1}} = \prod_{\tau=0}^{T}\left(\frac{\frac{\sum_{i\in G_{\tau,0}} q_i^\tau k p_i^0}{\sum_{i\in G_{\tau,0}} v_i q_i^\tau}}{\frac{\sum_{i\in G_{\tau,0}} q_i^\tau p_i^0}{\sum_{i\in G_{\tau,0}} v_i q_i^\tau}}\right)^{\frac{1}{T+1}} = (k^{T+1})^{\frac{1}{T+1}} = k.$$

Homogeneity in quantities

By rescaling the quantities in some s-th period, we transform a matrix of quantities Q into the new matrix Q^s, a vector of quality-adjusting factors v into the new vector v^s and we obtain that

$$P_{GEKS-AQU}^{0,t}(P, q^0, ..., kq^s, ..., q^t) = \left(\frac{\frac{\sum_{i\in G_{s,t}} kq_i^s p_i^t}{\sum_{i\in G_{s,t}} v_i^s kq_i^s}}{\frac{\sum_{i\in G_{s,0}} kq_i^s p_i^0}{\sum_{i\in G_{s,0}} v_i^s kq_i^s}}\right)^{\frac{1}{T+1}} \times \prod_{\tau=0,\tau\neq s}^{T}\left(\frac{\frac{\sum_{i\in G_{\tau,t}} q_i^\tau p_i^t}{\sum_{i\in G_{\tau,t}} v_i^s q_i^\tau}}{\frac{\sum_{i\in G_{\tau,0}} q_i^\tau p_i^0}{\sum_{i\in G_{\tau,0}} v_i^s q_i^\tau}}\right)^{\frac{1}{T+1}} =$$

$$= \left(\frac{\frac{\sum_{i\in G_{s,t}} q_i^s p_i^t}{\sum_{i\in G_{s,t}} v_i^s q_i^s}}{\frac{\sum_{i\in G_{s,0}} q_i^s p_i^0}{\sum_{i\in G_{s,0}} v_i^s q_i^s}}\right)^{\frac{1}{T+1}} \times \prod_{\tau=0,\tau\neq s}^{T}\left(\frac{\frac{\sum_{i\in G_{\tau,t}} q_i^\tau p_i^t}{\sum_{i\in G_{\tau,t}} v_i^s q_i^\tau}}{\frac{\sum_{i\in G_{\tau,0}} q_i^\tau p_i^0}{\sum_{i\in G_{\tau,0}} v_i^s q_i^\tau}}\right)^{\frac{1}{T+1}} = \prod_{\tau=0}^{T}\left(\frac{\frac{\sum_{i\in G_{\tau,t}} q_i^\tau p_i^t}{\sum_{i\in G_{\tau,t}} v_i^s q_i^\tau}}{\frac{\sum_{i\in G_{\tau,0}} q_i^\tau p_i^0}{\sum_{i\in G_{\tau,0}} v_i^s q_i^\tau}}\right)^{\frac{1}{T+1}}.$$

From the assumption that $G_0 = G_t$ we conclude that $G_{\tau,0} = G_{\tau,t}$ and also

$$P^{0,t}_{GEKS-AQU}(P,Q^s) = \prod_{\tau=0}^{T}\left(\frac{\frac{\sum_{i\in G_{\tau,t}} q_i^\tau p_i^t}{\sum_{i\in G_{\tau,t}} v_i^s q_i^\tau}}{\frac{\sum_{i\in G_{\tau,t}} q_i^\tau p_i^0}{\sum_{i\in G_{\tau,t}} v_i^s q_i^\tau}}\right)^{\frac{1}{T+1}} = \prod_{\tau=0}^{T}\left(\frac{\frac{\sum_{i\in G_{\tau,t}} q_i^\tau p_i^t}{\sum_{i\in G_{\tau,t}} v_i q_i^\tau}}{\frac{\sum_{i\in G_{\tau,t}} q_i^\tau p_i^0}{\sum_{i\in G_{\tau,t}} v_i q_i^\tau}}\right)^{\frac{1}{T+1}} =$$

$$= \prod_{\tau=0}^{T}\left(\frac{\frac{\sum_{i\in G_{\tau,t}} q_i^\tau p_i^t}{\sum_{i\in G_{\tau,t}} v_i q_i^\tau}}{\frac{\sum_{i\in G_{\tau,0}} q_i^\tau p_i^0}{\sum_{i\in G_{\tau,0}} v_i q_i^\tau}}\right)^{\frac{1}{T+1}} = P^{0,t}_{GS-GEKS}(P,Q).$$

Homogeneity in prices

By rescaling prices in the current period t, we transform a matrix of prices P into the new matrix P^t, a vector of quality-adjusting factors v into the new vector v^t and we obtain that

$$P^{0,t}_{GEKS-AQU}(p^0,...,kp^t,Q) = \prod_{\tau=0}^{T}\left(\frac{\frac{\sum_{i\in G_{\tau,t}} q_i^\tau k p_i^t}{\sum_{i\in G_{\tau,t}} v_i^t q_i^\tau}}{\frac{\sum_{i\in G_{\tau,0}} q_i^\tau p_i^0}{\sum_{i\in G_{\tau,0}} v_i^t q_i^\tau}}\right)^{\frac{1}{T+1}} =$$

$$= (k^{T+1})^{\frac{1}{T+1}} \times \prod_{\tau=0}^{T}\left(\frac{\frac{\sum_{i\in G_{\tau,t}} q_i^\tau p_i^t}{\sum_{i\in G_{\tau,t}} v_i^t q_i^\tau}}{\frac{\sum_{i\in G_{\tau,0}} q_i^\tau p_i^0}{\sum_{i\in G_{\tau,0}} v_i^t q_i^\tau}}\right)^{\frac{1}{T+1}} = k \times \prod_{\tau=0}^{T}\left(\frac{\frac{\sum_{i\in G_{\tau,t}} q_i^\tau p_i^t}{\sum_{i\in G_{\tau,t}} v_i^t q_i^\tau}}{\frac{\sum_{i\in G_{\tau,0}} q_i^\tau p_i^0}{\sum_{i\in G_{\tau,0}} v_i^t q_i^\tau}}\right)^{\frac{1}{T+1}}.$$

From the assumption that $G_0 = G_t$ we conclude that $G_{\tau,0} = G_{\tau,t}$ and also

$$P^{0,t}_{GEKS-AQU}(P^t,Q) = k \times \prod_{\tau=0}^{T}\left(\frac{\frac{\sum_{i\in G_{\tau,t}} q_i^\tau p_i^t}{\sum_{i\in G_{\tau,t}} v_i^t q_i^\tau}}{\frac{\sum_{i\in G_{\tau,t}} q_i^\tau p_i^t}{\sum_{i\in G_{\tau,t}} v_i^t q_i^\tau}}\right)^{\frac{1}{T+1}} = k \times \prod_{\tau=0}^{T}\left(\frac{\frac{\sum_{i\in G_{\tau,t}} q_i^\tau p_i^t}{\sum_{i\in G_{\tau,t}} v_i q_i^\tau}}{\frac{\sum_{i\in G_{\tau,t}} q_i^\tau p_i^t}{\sum_{i\in G_{\tau,t}} v_i q_i^\tau}}\right)^{\frac{1}{T+1}} =$$

$$= k \times \prod_{\tau=0}^{T}\left(\frac{\frac{\sum_{i\in G_{\tau,t}} q_i^\tau p_i^t}{\sum_{i\in G_{\tau,t}} v_i q_i^\tau}}{\frac{\sum_{i\in G_{\tau,0}} q_i^\tau p_i^t}{\sum_{i\in G_{\tau,0}} v_i q_i^\tau}}\right)^{\frac{1}{T+1}} = k \times P^{0,t}_{GEKS-AQU}(P,Q).$$

Commensurability

Let us also notice that the $P^{0,t}_{GEKS-AQU}$ index fulfils the weak version of the *commensurability* test. In fact, by rescaling $\tilde{p}_i^s = \lambda p_i^s$ and $\tilde{q}_i^s = \frac{q_i^s}{\lambda}$ for all $s \in [0,T]$ and $i \in G_s$ ($\lambda > 0$), we obtain a new vector of quality adjusting factors $v^\lambda = \lambda v$ and

$$P^{0,t}_{GEKS-AQU}(\tilde{P},\tilde{Q}) = \prod_{\tau=0}^{T}\left(\frac{\frac{\sum_{i\in G_{\tau,t}} \frac{\lambda}{\lambda} q_i^\tau p_i^t}{\sum_{i\in G_{\tau,t}} \lambda v_i^\lambda q_i^\tau}}{\frac{\sum_{i\in G_{\tau,0}} \frac{\lambda}{\lambda} q_i^\tau p_i^0}{\sum_{i\in G_{\tau,0}} \lambda v_i^\lambda q_i^\tau}}\right)^{\frac{1}{T+1}} = \prod_{\tau=0}^{T}\left(\frac{\frac{\sum_{i\in G_{\tau,t}} q_i^\tau p_i^t}{\sum_{i\in G_{\tau,t}} \lambda v_i q_i^\tau}}{\frac{\sum_{i\in G_{\tau,0}} q_i^\tau p_i^0}{\sum_{i\in G_{\tau,0}} \lambda v_i q_i^\tau}}\right)^{\frac{1}{T+1}} = P^{0,t}_{GEKS-AQU}(P,Q).$$

Please note, that if $G_0 = G_t$, the GEKS-AQU index satisfies the full (strong) version of the *commensurability* test.